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Systems of Revolution and their Relation to Conical Systems, in the Theory of Lamé's Products.

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SECTION I.

*Introduction.**

The problem treated in the first part of the present paper is an application of a theorem due to Lord Kelvin, by means of which, from a known solution, V, of Laplace's Equation in terms of coördinates corresponding to a system of mutually orthogonal surfaces, a solution may be readily deduced for a new system of surfaces obtained from the first by inversion.

This theorem has been used in an extended sense, so that real surfaces have been obtained from imaginary surfaces by inversions with regard to imaginary points as centres.

Definition of Lamé's Products.

In problems requiring the solution of Laplace's Equation, it is often possible to obtain a solution by transferring to curvilinear coördinates, λ, μ, ν, and assuming that V is a product of three factors, i. e.

$$V = L \,.\, M \,.\, N, \tag{1}$$

where L, M, N are functions of λ, μ, ν respectively. Such an expression for V is called a Lamé's Product.

In cases where a solution in the form of (1) cannot be obtained, it is sometimes possible to obtain a solution on the assumption that

$$V = T \,.\, L \,.\, M \,.\, N, \tag{2}$$

where L, M, N have the same meaning as before. But T is a factor which is a function of more than one of the quantities λ, μ, ν, and, moreover, is independent of the accessory parameters which occur in L, M, N. These parameters first appear when the new form of Laplace's Equation is resolved into ordinary differential equations.

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*The writer gratefully acknowledges his indebtedness to Professor Maxime Böcher, of Harvard University, for suggestions and criticisms.

Several topics in the first section of this paper have been discussed by A. Wangerin with the aid of elliptic functions. In the following pages the use of elliptic functions has been avoided, while a simpler and more symmetrical notation than is ordinarily used by writers on this subject has been adopted. Wangerin's earliest paper* gives the ordinary differential equations obtained when the extraneous factor is $\frac{1}{\sqrt{r}}$. Another paper by the same author will be mentioned later.

Theorem of Lord Kelvin.

The theorem of Lord Kelvin, before referred to, shows that if $V(xyz)$ is a solution of Laplace's Equation, then

$$\frac{1}{\sqrt{x^2+y^2+z^2}}\cdot V\left[\frac{\rho x}{x^2+y^2+z^2},\ \frac{\rho y}{x^2+y^2+z^2},\ \frac{\rho z}{x^2+y^2+z^2}\right] \tag{3}$$

is also a solution. Here space has been subjected to an inversion with regard to a sphere of radius $\sqrt{\rho}$ described about the origin.

If then there are given three families of mutually orthogonal surfaces,

$$F_1(xyz)=\lambda\,, \tag{4}$$
$$F_2(xyz)=\mu\,, \tag{5}$$
$$F_3(xyz)=\nu\,, \tag{6}$$

and if in terms of these curvilinear coördinates λ, μ, ν a solution of Laplace's Equation be obtained in the form of a Lamé's Product, then, by applying Lord Kelvin's theorem, Lamé's Products (with an extraneous factor) may be at once obtained for the system of surfaces derived from (4)(5)(6) by inversion.

Surfaces of Revolution obtained from Cones of the Second Degree.

The starting point here is a system of confocal cones of the second degree and concentric spheres.

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[*] Reduction der Potentialgleichung. Preisschriften der Fürstlich Jablnowskischen Gesellschaft der Wissenschaften. No. 18. Leipzig, 1875.

This primary system of surfaces will be transformed by inversion into a system of revolution.

Then since Lamé's products

$$V = L \, . \, M \, . \, N \tag{7}$$

can be found for the cone system, solutions of Laplace's Equation may be deduced from them for the new system in the form

$$V = T \, . \, L \, . \, M \, . \, N, \tag{8}$$

where T is an extraneous factor briefly treated above. The first topic is the transformation of the cone system into a system of revolution. The equations of the cone system may be written

$$x^2 + y^2 + z^2 = \lambda, \tag{9}$$

$$\frac{x^2}{\mu - \varepsilon_1} + \frac{y^2}{\mu - \varepsilon_2} + \frac{z^2}{\mu - \varepsilon_3} = 0, \tag{10}$$

$$\frac{x^2}{\nu - \varepsilon_1} + \frac{y^2}{\nu - \varepsilon_2} + \frac{z^2}{\nu - \varepsilon_3} = 0. \tag{11}$$

In order to obtain from this system a system of revolution, it is necessary and sufficient that the concentric spheres shall be transformed into meridian planes.

An inversion of the family of concentric spheres with regard to any point except their centre will give a family of eccentric spheres, having a common circle of intersection, real or imaginary. A second inversion with regard to a point on the circle of intersection will transform this circle into a right line. This right line is, accordingly, the axis of the planes into which the eccentric spheres have been transformed.

The explicit formulæ of transformation will now be obtained by means of a detailed transformation of the concentric spheres. It will simplify the formulæ for an inversion if the origin be first changed to the chosen centre of inversion.

The centre of the first inversion will be taken at any point in space (l, m, n) where all three coördinates are not zero.

The formulæ of transformation are

$$x = x_1 + l, \quad y = y_1 + m, \quad z = z_1 + n. \tag{12}$$

The family of concentric spheres (9) is now

$$(x_1 + l)^2 + (y_1 + m)^2 + (z_1 + n)^2 = \lambda. \tag{13}$$

The next transformation is an inversion with regard to a unit sphere whose centre is at the origin, i. e.

$$x_1 = \frac{x_2}{x_2^2 + y_2^2 + z_2^2} \quad y_1 = \frac{y_2}{x_2^2 + y_2^2 + z_2^2} \quad z_1 = \frac{z_2}{x_2^2 + y_2^2 + z_2^2}. \tag{14}$$

The result is the following family of eccentric spheres:

$$(x_2^2 + y_2^2 + z_2^2)(l^2 + m^2 + n^2 - \lambda) + 2lx_2 + 2my_2 + 2nz_2 + 1 = 0. \tag{15}$$

The centre for the second inversion is the point (a, b, c), which is to be taken on the circle of intersection of the eccentric spheres.

It follows that a, b, c satisfy (15) for all values of λ, giving

$$a^2 + b^2 + c^2 = 0, \tag{16}$$

$$2al + 2bm + 2cn + 1 = 0. \tag{17}$$

The equations for the change of origin are

$$x_2 = x_3 + a \quad y_2 = y_3 + b \quad z_2 = z_3 + c. \tag{18}$$

By the use of (16)(17)(18) the equation of the eccentric spheres (15) becomes

$$(x_3^2 + y_3^2 + z_3^2 + 2ax_3 + 2by_3 + 2cz_3)(l^2 + m^2 + n^2 - \lambda) + 2lx_3 + 2my_3 + 2nz_3 = 0. \tag{19}$$

The second inversion will be made with regard to a sphere whose radius is $\sqrt{\rho}$, i. e.

$$x_3 = \frac{\rho x_4}{x_4^2 + y_4^2 + z_4^2} \quad y_3 = \frac{\rho y_4}{x_4^2 + y_4^2 + z_4^2} \quad z_3 = \frac{\rho z_4}{x_4^2 + y_4^2 + z_4^2}. \tag{20}$$

Equation (19) becomes

$$(2ax_4 + 2by_4 + 2cz_4 + \rho)(l^2 + m^2 + n^2 - \lambda) + 2(lx_4 + my_4 + nz_4) = 0. \tag{21}$$

Equation (21) represents a family of planes whose axis is the intersection of the following planes:

$$2ax_4 + 2by_4 + 2cz_4 + \rho = 0, \tag{22}$$

$$lx_4 + my_4 + nz_4 = 0. \tag{23}$$

It will be simpler to choose the coefficients in (22) and (23), so that the axis of the family of planes shall be perpendicular to the $y_4 z_4$ plane. The condition equation is

$$l = a = 0. \tag{24}$$

The origin is next transferred to a point (d, e, f) taken on the axis at its intersection with the $y_4 z_4$ plane. Thence come the condition equations

$$2ad + 2be + 2cf + \rho = 0\,, \tag{25}$$

$$ld + me + nf = 0\,, \tag{26}$$

$$d = 0\,. \tag{27}$$

The equations for the change of origin are

$$x_4 = x_5 + d \qquad y_4 = y_5 + e \qquad z_4 = z_5 + f. \tag{28}$$

By the use of (24) . . . (28) the family of planes, (21), takes the final form, omitting subscripts,

$$y - z \tan \phi = 0\,, \tag{29}$$

$$\tan \phi = \frac{c\,(1 - 4b^2\lambda)}{b\,(1 + 4b^2\lambda)}\,. \tag{30}$$

It will simplify several of the later equations and involve no loss in generality to assume the equation

$$4b^2 - 1 = 0. \tag{31}$$

The transformation and condition equations may now be reduced to the form

$$4b^2 = -\,4c^2 = 1\,, \tag{32}$$

$$x = \frac{x_5}{2\,(by_5 + cz_5)}\,, \tag{33}$$

$$y = \frac{b}{2\rho\,(by_5 + cz_5)}\,(x_5^2 + y_5^2 + z_5^2 - \rho^2)\,, \tag{34}$$

$$z = \frac{c}{2\rho\,(by_5 + cz_5)}\,(x_5^2 + y_5^2 + z_5^2 + \rho^2)\,, \tag{35}$$

$$x^2 + y^2 + z^2 = \frac{-\,(by_5 - cz_5)}{(by_5 + cz_5)}\,. \tag{36}$$

The inverse of the above transformations will be of use later and is

$$x^5 = \frac{-\,\rho x}{2\,(by + cz)}\,, \tag{37}$$

$$y_5 = \frac{\rho b}{2\,(by + cz)}\,(x^2 + y^2 + z^2 - 1)\,, \tag{38}$$

$$z_5 = \frac{\rho c}{2\,(by + cz)}\,(x^2 + y^2 + z^2 + 1)\,, \tag{39}$$

$$x_5^2 + y_5^2 + z_5^2 = -\,\frac{\rho^2\,(by - cz)}{(by + cz)}\,. \tag{40}$$

The transformations defined by (32) . . . (36) have been shown to change the concentric spheres of (9) into meridian planes. Applying the same transformations to the cones of (10) and (11) gives, omitting subscripts,

$$\frac{4\rho^2x^2}{\mu-\varepsilon_1}+\frac{(x^2+y^2+z^2-\rho^2)^2}{\mu-\varepsilon_2}-\frac{(x^2+y^2+z^2+\rho^2)^2}{\mu-\varepsilon_3}=0, \tag{41}$$

and a similar equation with μ replaced by ν.

Equations (29) and (41) define a system of revolution obtained from the primary system of cones and spheres.

Rotation Cyclids.

Equation (41) defines a class of surfaces known as cyclids of rotation,* and may be thrown into a more familiar form by a linear transformation of μ, ε_1, ε_2, ε_3 defined by the equations

$$\mu=\frac{\alpha\bar{\mu}+\beta}{\gamma\bar{\mu}+\delta} \quad \gamma\bar{\varepsilon}_2+\delta=0, \tag{42}$$

$$\varepsilon_1=\frac{\alpha\bar{\varepsilon}_1+\beta}{\gamma\bar{\varepsilon}_1+\delta}. \tag{43}$$

By transformation of the same type as (43), ε_2 and ε_3 are replaced by $\bar{\varepsilon}_3$ and $\bar{\varepsilon}_4$ respectively.

The result is

$$\frac{4\rho^2x^2}{\bar{\mu}-\bar{\varepsilon}_1}+\frac{4\rho^2(y^2+z^2)}{\bar{\mu}-\bar{\varepsilon}_2}+\frac{(x^2+y^2+z^2-\rho^2)^2}{\bar{\mu}-\bar{\varepsilon}_3}-\frac{(x^2+y^2+z^2+\rho^2)^2}{\bar{\mu}-\varepsilon_4}=0. \tag{44}$$

In a later section it will be necessary to have the values of x^2 and $r^2=y^2+z^2$ obtained from (44) and the similar equation where ν replaces μ. The direct solution is possible but extremely long, so that it is more convenient to proceed in the following manner.

*** The possibility of obtaining cyclids of rotation in the manner shown appears from the table on p. 65 in Professor Böcher's book "Ueber die Reihenentwickelungen der Potentialtheorie," Taubner, Leipzig, 1894.**

From (9)(10)(11),

$$x^2 = \frac{\lambda\,(\mu - \varepsilon_1)(\nu - \varepsilon_1)}{(\varepsilon_2 - \varepsilon_1)(\varepsilon_3 - \varepsilon_1)}, \tag{45}$$

$$y^2 = \frac{\lambda\,(\mu - \varepsilon_2)(\nu - \varepsilon_2)}{(\varepsilon_3 - \varepsilon_2)(\varepsilon_1 - \varepsilon_2)}, \tag{46}$$

$$z^2 = \frac{\lambda\,(\mu - \varepsilon_3)(\nu - \varepsilon_3)}{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}. \tag{47}$$

From (32)(37) . . . (40),

$$x_5^2 = \frac{\rho^2 x^2}{(y \pm zi)^2}, \tag{48}$$

$$r_5^2 = y_5^2 + z_5^2 = -\frac{\rho^2\,(x^2 + y^2 + z^2)}{(y \pm zi)^2}. \tag{49}$$

Since the coördinate axes of the cyclid system are the $x_5 y_5 z_5$ axes, it is necessary merely to express x_5^2 and r_5^2 in terms of μ and ν by means of (42)(43)(45)(46)(47). The result is, dropping dashes and subscripts,

$$x_2 = \frac{\rho^2\,(\mu - \varepsilon_1)(\nu - \varepsilon_1)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)}{(\varepsilon_1 - \varepsilon_2)\left[\sqrt{(\mu - \varepsilon_3)(\nu - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)} \pm \sqrt{(\mu - \varepsilon_4)(\nu - \varepsilon_4)(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}\right]^2}, \tag{50}$$

$$r^2 = \frac{\rho^2\,(\mu - \varepsilon_2)(\nu - \varepsilon_2)(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)}{(\varepsilon_2 - \varepsilon_1)\left[\sqrt{(\mu - \varepsilon_3)(\nu - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)} \pm \sqrt{(\mu - \varepsilon_4)(\nu - \varepsilon_4)(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)}\right]^2}. \tag{51}$$

*Laplace's Equation in conical coördinates.**

It is now desired to obtain V of Laplace's Equation in the form of Lamé's Products and an extraneous factor, the parameters in V corresponding to the surfaces of the system of revolution (29)(41). For this purpose it is necessary to obtain V first for the cone system in the form of Lamé's Products.

From (45)(46)(47)

$$ds^2 = \tfrac{1}{4}\left[\frac{d\lambda^2}{\lambda} + \frac{\lambda\,(\nu - \mu)\,d\mu^2}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)} + \frac{\lambda\,(\mu - \nu)\,d\nu^2}{(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)}\right]. \tag{52}$$

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* The results stated in this section are familiar, although usually given in a somewhat different notation. See Lamé's original memoir in Liouville's Journal, vol. 4, p. 137.

In place of λ, μ, ν are to be introduced u, v, w, defined as follows:

$$u = \int \frac{d\lambda}{2\sqrt{\lambda^3}}, \tag{53}$$

$$v = \int \frac{d\mu}{2\sqrt{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)}}, \tag{54}$$

$$w = \int \frac{d\nu}{2\sqrt{(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)}}. \tag{55}$$

Equation (52) becomes

$$ds^2 = \lambda^2 du^2 + \lambda(\nu - \mu)\, dv^2 + \lambda(\mu - \nu)\, dw^2. \tag{56}$$

Laplace's Equation in curvilinear coördinates u, v, w is

$$D_u\left(\frac{h_1}{h_2 h_3} D_u V\right) + D_v\left(\frac{h_2}{h_3 h_1} D_v V\right) + D_w\left(\frac{h_3}{h_1 h_2} D_w V\right) = 0. \tag{57}$$

The values of h_1^2, h_2^2, h_3^2 are the reciprocals of the coefficients of du^2, dv^2, dw^2 respectively in (56).

Equation (57) is now written

$$\frac{1}{\lambda} D_u^2 V + \frac{1}{\nu - \mu} D_v^2 V + \frac{1}{\mu - \nu} D_w^2 V = 0. \tag{58}$$

Letting

$$V = L.M.N, \tag{59}$$

(58) is resolved by the usual method into the three equations

$$\frac{d^2L}{du^2} - (A\lambda)L \qquad = 0, \tag{60}$$

$$\frac{d^2M}{dv^2} - (A\mu + B)M = 0, \tag{61}$$

$$\frac{d^2N}{dw^2} - (A\nu + B)N = 0. \tag{62}$$

The general solution of (60) is

$$L = A_1\lambda^{\frac{m}{2}} + B_1\lambda^{-\frac{m+1}{2}}, \tag{63}$$

where m is thus defined:

$$A = m(m+1). \tag{64}$$

Equations (61) and (62) are Lamé's Equations, whose solutions may be indicated thus:

$$M = E_1(\mu), \quad N = E_2(\nu). \tag{65}$$

It follows that (58) has solutions in the form of Lamé's Products, i. e.

$$V = L.M.N. = (A_1\lambda^{\frac{m}{2}} + B_1\lambda^{-\frac{m+1}{2}})\,E_1(\mu)\,E_2(\nu). \tag{66}$$

Application of the Theorem of Lord Kelvin.

Since (66) gives a value of V in terms of parameters corresponding to the surfaces of the cone system, the value of V for the system of rotation, (29)(41) is written at once by the theorem of Lord Kelvin,

$$V = T.(A_1\lambda^{\frac{m}{2}} + B_1\lambda^{-\frac{m+1}{2}})\,E_1(\mu)\,E_2(\nu). \tag{67}$$

The parameters λ, μ, ν correspond now to the surfaces of the system of rotation.

It remains to calculate the factor T, and this is most easily done by the aid of the transformation formulæ in the detailed form given by (12)(14)(16)(17)(18) (20)(24) . . . (28)(31).

Starting with (59) the first inversion gives

$$V = \frac{1}{\sqrt{x_2^2 + y_2^2 + z_2^2}} \,.\, L.M.N, \tag{68}$$

while the result of the other changes is

$$V = \frac{1}{\sqrt{2\rho\,(by_5 + cz_5)}} \,.\, L.M.N. \tag{69}$$

Equation (69) may be improved in form by replacing the parameter λ by the angle between any meridian plane and the plane $x_5 z_5$. This angle has been denoted by ϕ, and from (30)(31)

$$\lambda = (\cos\phi \pm i\sin\phi)^2. \tag{70}$$

Also let R be the distance from the axis of rotation to any point $(x_5\, y_5\, z_5)$, then it follows that

$$y_5 = R\sin\phi, \quad z_5 = R\cos\phi, \tag{71}$$

$$\frac{1}{\sqrt{2\rho\,(by_5 + cz_5)}} = \frac{1}{\sqrt{R}} \cdot \frac{1}{\sqrt{2\rho\,(b\sin\phi + c\cos\phi)}}$$

$$= \frac{c_0}{\sqrt{R}}(\cos\phi \pm i\sin\phi)^{\frac{1}{2}}. \tag{72}$$

Making in (69) the substitutions indicated in (70)(72) and giving to L, M, N the values in (66), the result is

$$V = \frac{1}{\sqrt{R}} \left[A_0 \cos\left(m + \tfrac{1}{2}\right)\phi + B_0 \sin\left(m + \tfrac{1}{2}\right)\phi \right] E_1(\mu)\, E_2(\nu). \tag{73}$$

Equation (73) gives the desired value of V in the form of Lamé's Products and an extraneous factor, the parameters ϕ, μ, ν corresponding to the surfaces of the system of revolution derived from the cone system, as previously explained. The extraneous factor involves R only, i. e. the distance of any point from the axis of revolution.

The accessory parameters in $E_1(\mu)$ and $E_2(\nu)$ are A, which equals $m(m+1)$, and B.

The most general system of revolution obtainable from the given cone system.

Returning to the system of revolution, (29)(41), it is evident that any further transformation which gives a rotation system must leave the axis unchanged. It is then expressible as an inversion with regard to a point on the axis, accompanied by a translation along the axis, the equations being

$$x = \frac{a(\bar{x} + q)}{(\bar{x} + q)^2 + \bar{y}^2 + \bar{z}^2} + p, \quad y = \frac{a\bar{y}}{(\bar{x} + q)^2 + \bar{y}^2 + \bar{z}^2}, \quad z = \frac{a\bar{z}}{(\bar{x} + q)^2 + \bar{y}^2 + \bar{z}^2}. \tag{74}$$

The resulting system of revolution is the most general type obtainable from the cones of the second degree defined by (10) and (11).

The same result may be reached in a different way by letting the axis of revolution make the angles α_1, β_1, γ_1, with the x, y, z axes in (9)(10)(11), and then making the transformations defined by (32) . . . (36).

The following relations will be useful for reference:

$$\cos\alpha_1 = \frac{2pq + a}{a}, \tag{75}$$

$$\cos\beta_1 = \frac{p^2 q + pa - \rho^2 q}{\rho a}, \tag{76}$$

$$\cos\gamma_1 = \frac{\pm i(p^2 q + pa + \rho^2 q)}{\rho a}, \tag{77}$$

and conversely

$$p = \frac{\rho(1 - \cos\alpha_1)}{\cos\beta_1 \pm i\cos\gamma_1}, \tag{78}$$

$$q = -\frac{a}{2\rho}(\cos\beta_1 \pm i\cos\gamma_1). \tag{79}$$

SECTION II.

Determination of the most general surfaces of revolution generated by confocal cyclic curves, for which Lamé's Products, with the extraneous factor $\frac{1}{\sqrt{r}}$, exist.

In this section will be considered surfaces of revolution whose meridian curves are obtained from those of the symmetric cyclids by circular transformation. It will be convenient to express these circular transformations as linear transformations in the complex plane.

Laplace's Equation will be obtained in curvilinear coördinates corresponding to the families of the system, and it will be shown that in certain special cases an expression for V may be found in the form of Lamé's Products, with an extraneous factor.

The equations of the meridian curves of the symmetric cyclids are

$$\frac{4\rho^2x^2}{\mu - \varepsilon_1} + \frac{4\rho^2r^2}{\mu - \varepsilon_2} + \frac{(x^2 + r^2 - \rho^2)^2}{\mu - \varepsilon_3} - \frac{(x^2 + r^2 + \rho^2)^2}{\mu - \varepsilon_4} = 0, \tag{1}$$

$$\frac{4\rho^2x^2}{\nu - \varepsilon_1} + \frac{4\rho^2r^2}{\nu - \varepsilon_2} + \frac{(x^2 + r^2 - \rho^2)^2}{\nu - \varepsilon_3} - \frac{(x^2 + r^2 + \rho^2)^2}{\nu - \varepsilon_4} = 0. \tag{2}$$

The general linear transformation in the complex plane is

$$x' + r'i = \frac{\alpha + \beta(x + ri)}{\gamma + \delta(x + ri)}, \tag{3}$$

from which follows:

$$x' - r'i = \frac{\alpha_1 + \beta_1(x - ri)}{\gamma_1 + \delta_1(x - ri)}. \tag{4}$$

The constants α, β, γ, δ are not specialized, while α_1, β_1, γ_1, δ_1 are their respective conjugates.

In the previous section, (50) and (51) give the values of x^2 and r^2 in terms of μ and ν.

Let θ be the angle between any meridian plane and the $x'\,y'$ plane, then the relations between y', z', r', θ are

$$y' = r' \cos\theta, \quad z' = r' \sin\theta. \tag{5}$$

Thus the connection has been established between the rectangular coördinates x', y', z' and the curvilinear coördinates μ, ν, θ of the surfaces forming the basis of the present paper.

Laplace's Equation in coördinates μ, ν, θ and a condition for its resolution.

Laplace's Equation in curvilinear coördinates has been given in the previous section, (57).

From (5),

$$\frac{1}{h_1^2} = (D_\mu x')^2 + (D_\mu r')^2, \tag{6}$$

$$\frac{1}{h_2^2} = (D_\nu x')^2 + (D_\nu r')^2, \tag{7}$$

$$\frac{1}{h_3^2} = 1. \tag{8}$$

Assume that V has the form

$$V = \frac{1}{\sqrt{r'}}\, \mathrm{M}\,.\,\mathrm{N}\,.\,\Theta, \tag{9}$$

where M, N, Θ are functions of μ, ν, θ respectively. Applying (8) and (9) to Laplace's Equation in curvilinear coördinates and noticing that h_1 and h_2 are independent of θ, gives

$$h_1\, h_2\, r'^{\frac{3}{2}} \left[\frac{D_\mu \left(\frac{h_1 r'}{h_2} D_\mu \frac{M}{\sqrt{r'}} \right)}{M} + \frac{D_\nu \left(\frac{h_2 r'}{h_1} D_\nu \frac{N}{\sqrt{r'}} \right)}{N} \right] + \frac{D_\theta^2 \Theta}{\Theta} = 0. \tag{10}$$

Equation (10) may be resolved into two equations, viz.

$$D_\theta^2 \Theta + m^2\, \Theta = 0, \tag{11}$$

$$D_\mu \frac{\left(\frac{h_1 r'}{h_2} D_\mu \frac{M}{\sqrt{r'}} \right)}{M} + D_\nu \frac{\left(\frac{h_2 r'}{h_1} D_\nu \frac{N}{\sqrt{r'}} \right)}{N} - \frac{m^2}{h_1\, h_2\, r'^{\frac{3}{2}}} = 0. \tag{12}$$

The general solution of (11) is

$$\Theta = A \cos m\,\theta + B \sin m\,\Theta. \tag{13}$$

In order to simplify (12), it is necessary to calculate the value of $\frac{h_1}{h_2}$.

From (3) and (4) come four equations of the form

$$D_\mu x' + iD_\mu r' = \frac{(\beta\gamma - \alpha\delta)(D_\mu x + iD_\mu r)}{(\gamma + \delta(x + ir))^2}. \tag{14}$$

From (6)(7) and (14),

$$\frac{h_2^2}{h_1^2} = \frac{(D_\mu x')^2 + (D_\mu r')^2}{(D_\nu x')^2 + (D_\nu r')^2} = \frac{(D_\mu x)^2 + (D_\mu r)^2}{(D_\nu x)^2 + (D_\nu r)^2}. \tag{15}$$

Equation (15) requires the calculation of four expressions of the type $(D_\mu x)^2$. From the value of x^2 referred to above

$$\frac{4}{\rho^2} \cdot (D_\mu x)^2 = \frac{(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)}{(\varepsilon_1 - \varepsilon_2)} \cdot \tag{16}$$

$$\frac{\left[\begin{aligned}&(\varepsilon_1 - \varepsilon_3)\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_2)(\mu - \varepsilon_4)(\nu - \varepsilon_1)(\nu - \varepsilon_3)} \\ &\qquad \pm (\varepsilon_1 - \varepsilon_4)\sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\nu - \varepsilon_1)(\nu - \varepsilon_4)}\end{aligned}\right]^2}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\mu - \varepsilon_4)\left[\begin{aligned}&\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_3)(\nu - \varepsilon_3)} \\ &\qquad \pm \sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_4)(\nu - \varepsilon_4)}\end{aligned}\right]^4}.$$

Interchanging ε_1 and ε_2 in (16) gives $(D_\mu r)^2$. From these two values an interchange of μ and ν gives $(D_\nu x)^2$ and $(D_\nu r)^2$.

From the four expressions above mentioned,

$$(D_\mu x)^2 + (D_\mu r)^2 =$$

$$\frac{\rho^2}{4} \cdot \frac{\dfrac{(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)(\nu - \mu)}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\mu - \varepsilon_4)}}{\left[\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_3)(\nu - \varepsilon_3)} \pm \sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_4)(\nu - \varepsilon_4)}\right]^2}, \tag{17}$$

$$(D_\nu x)^2 + (D_\nu r)^2 =$$

$$\frac{\rho^2}{4} \cdot \frac{\dfrac{(\varepsilon_1 - \varepsilon_3)(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_3)(\varepsilon_2 - \varepsilon_4)(\varepsilon_3 - \varepsilon_4)(\mu - \nu)}{(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)(\nu - \varepsilon_4)}}{\left[\sqrt{(\varepsilon_1 - \varepsilon_4)(\varepsilon_2 - \varepsilon_4)(\mu - \varepsilon_3)(\nu - \varepsilon_3)} \pm \sqrt{(\varepsilon_1 - \varepsilon_3)(\varepsilon_2 - \varepsilon_3)(\mu - \varepsilon_4)(\nu - \varepsilon_4)}\right]}. \tag{18}$$

By the aid of (17)(18) equation (15) becomes

$$\frac{h_2^2}{h_1^2} = \frac{-(\nu - \varepsilon_1)(\nu - \varepsilon_2)(\nu - \varepsilon_3)(\nu - \varepsilon_4)}{(\mu - \varepsilon_1)(\mu - \varepsilon_2)(\mu - \varepsilon_3)(\mu - \varepsilon_4)}. \tag{19}$$

Assume two new parameters s and t defined by the equations

$$s = \int \frac{d\mu}{2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)}}\,, \tag{20}$$

$$t = \int \frac{d\nu}{2\sqrt{(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_4)}}\,. \tag{21}$$

From (19) . . . (21)

$$\frac{h_2^2}{h_1^2} + \frac{\left(\frac{d\nu}{dt}\right)^2}{\left(\frac{d\mu}{ds}\right)^2} = 0. \tag{22}$$

Using (20) . . . (22), it is now possible to write (12) in a new form, viz.

$$\frac{D_s^2 M}{M} - \frac{D_t^2 N}{N} - \frac{1}{2r'}\left[D_s^2 r' - D_t^2 r'\right] + \frac{1}{4r'^2}\left[(D_s r')^2 - (D_t r')^2\right] \mp \frac{m^2 i}{r^2 h_1 h_2}\cdot\left(\frac{d\mu}{ds}\right)\cdot\left(\frac{d\nu}{dt}\right) = 0. \tag{23}$$

From (7),

$$\frac{1}{h_2^2} = (D_\nu x')^2 + (D_\nu r')^2 = \frac{(D_t x')^2 + (D_t r')^2}{\left(\frac{d\nu}{dt}\right)^2}\,. \tag{24}$$

The last term of (23) is reduced by the use of (22) and (24), thus:

$$\frac{\mp m^2 i}{r^2 h_1 h_1}\left(\frac{d\mu}{ds}\right)\left(\frac{d\nu}{dt}\right) = \frac{m^2}{r'^2}\left[(D_t x')^2 + (D_t r')^2\right]. \tag{25}$$

The third and fourth terms of (23) are reduced by the following calculations.

From (16) and three others of the same type, and with the aid of (20) and (21),

$$\frac{(D_\mu x)^2}{(D_\mu r)^2} = \frac{-(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_4)}{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)} = \frac{-\left(\frac{d\nu}{dt}\right)^2}{\left(\frac{d\mu}{ds}\right)^2}\,, \tag{26}$$

$$\frac{(D_\nu x)^2}{(D_\nu r)^2} = \frac{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)}{(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_4)} = \frac{-\left(\frac{d\mu}{ds}\right)^2}{\left(\frac{d\nu}{dt}\right)^2}\,. \tag{27}$$

Equations (26) and (27) give

$$(D_s x)^2 + (D_t r)^2 = 0, \tag{28}$$

$$(D_t x)^2 + (D_s r)^2 = 0. \tag{29}$$

The condition for orthogonal curves is

$$D_s x \,.\, D_t x + D_s r \,.\, D_t r = 0. \tag{30}$$

From (3) and (4) come four equations of the following type:

$$2D_s x' = \frac{(\beta\gamma - \alpha\delta)(D_s x + iD_s r)}{(\gamma + \delta(x + ir))^2} + \frac{(\beta_1\gamma_1 - \alpha_1\delta_1)(D_s x - iD_s r)}{(\gamma_1 + \delta_1(x - ir))^2}. \tag{31}$$

Equations (28) . . . (31) give

$$D_s x' = \pm\, iD_t r', \quad D_t x' = \pm\, iD_s r', \tag{32}$$

from which follow

$$D_s^2 r' - D_t^2 r' \quad = 0, \tag{33}$$

$$(D_s r)^2 + (D_t x')^2 = 0. \tag{34}$$

From (25)(33)(34) the final form of (23) is

$$\frac{D_s^2 M}{M} - \frac{D_t^2 N}{N} + (m^2 - \tfrac{1}{4})\,\frac{[(D_t x')^2 + (D_t r')^2]}{r'^2} = 0. \tag{35}$$

Since the first two terms of (35) are functions of s and t respectively, it follows that the third term must be the sum of two functions, one of s alone and the other of t alone. In that case the equation may be resolved into two ordinary differential equations. Hence the necessary and sufficient condition for the existence of a solution of Laplace's equation in the form

$$V = \frac{1}{\sqrt{r}} \,.\, \mathrm{M} \,.\, \mathrm{N} \,.\, \Theta \tag{36}$$

is the following:

$$\frac{(D_t x')^2 + (D_t r')^2}{r'^2} = H_1(s) + H_2(t). \tag{37}$$

Four cases resulting from the condition.

To determine the constants in the general linear transformation, (3), in such a way that (37) shall be satisfied, is the next important topic in this paper.

By the aid of (3) and (4) the first member of (37) becomes

$$\frac{(D_tx')^2+(D_tr')^2}{r'^2}=\frac{K[(D_tx)^2+(D_tr)^2]}{[A+E(x^2+r^2)+(B+D)x+(B-D)ir]^2}. \tag{38}$$

In (38) the following abbreviations are used:

$$A=\alpha\gamma_1-\alpha_1\gamma,\quad B=\beta\gamma_1-\beta_1\gamma, \tag{39}$$
$$D=\alpha\delta_1-\alpha_1\delta,\quad E=\beta\delta_1-\beta_1\delta, \tag{40}$$
$$K=-4(\alpha\delta-\beta\gamma)(\alpha_1\delta_1-\beta_1\gamma_1). \tag{41}$$

The numerator of (38) may be reduced by noticing that

$$(D_tx)^2+(D_tr)^2=[(D_vx)^2+(D_vr)^2]\left(\frac{dv}{dt}\right)^2. \tag{42}$$

From (18)(21)(42)

$$(D_tx)^2+(D_tr)=\frac{\rho^2(\varepsilon_1-\varepsilon_3)(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_3)(\varepsilon_2-\varepsilon_4)(\varepsilon_3-\varepsilon_4)(\mu-\nu)}{[\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)(\mu-\varepsilon_3)(\nu-\varepsilon_3)}\pm\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_4)}]^2}. \tag{43}$$

From the values of x^2 and r^2, after some reduction,

$$x^2+r^2=-\rho^2\frac{[\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)(\mu-\varepsilon_3)(\nu-\varepsilon_3)}\mp\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_4)}]}{\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)(\mu-\varepsilon_3)(\nu-\varepsilon_3)}\pm\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_4)}}. \tag{44}$$

Several new abbreviations are introduced as follows:

$$\bar{A}=\pm\rho(B+D)\sqrt{\frac{(\varepsilon_2-\varepsilon_3)(\varepsilon_2-\varepsilon_4)(\varepsilon_3-\varepsilon_4)}{(\varepsilon_1-\varepsilon_2)}}, \tag{45}$$
$$\bar{B}=\pm\rho(B-D)\sqrt{\frac{(\varepsilon_1-\varepsilon_3)(\varepsilon_1-\varepsilon_4)(\varepsilon_3-\varepsilon_4)}{(\varepsilon_1-\varepsilon_2)}}, \tag{46}$$
$$\bar{C}=(A-E\rho^2)\sqrt{(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_4)}, \tag{47}$$
$$\bar{D}=\pm(A+E\rho^2)\sqrt{(\varepsilon_1-\varepsilon_3)(\varepsilon_2-\varepsilon_3)}, \tag{48}$$
$$H=\rho^2(\varepsilon_1-\varepsilon_3)(\varepsilon_1-\varepsilon_4)(\varepsilon_2-\varepsilon_3)(\varepsilon_2-\varepsilon_4)(\varepsilon_3-\varepsilon_4). \tag{49}$$

None of the quantities introduced by (39) . . . (41)(45) . . . (49) involve μ or ν. By means of (43) . . . (49) a new form is obtained for (38), viz.

$$\frac{(D_tx')^2+(D_tr')^2}{r'^2}=\frac{H.K.(\mu-\nu)}{\left[\begin{array}{l}\bar{A}\sqrt{(\mu-\varepsilon_1)(\nu-\varepsilon_1)}+\bar{B}\sqrt{(\mu-\varepsilon_2)(\nu-\varepsilon_2)}\\ \quad+\bar{C}\sqrt{(\mu-\varepsilon_3)(\nu-\varepsilon_3)}+\bar{D}\sqrt{(\mu-\varepsilon_4)(\nu-\varepsilon_4)}\end{array}\right]^2}. \tag{50}$$

Let the second member of (50) be denoted by $H.K.\phi(\mu, \nu)$. Then from (37) it follows that values of $\bar{A}, \bar{B}, \bar{C}, \bar{D}$ are to be found such that $\phi(\mu, \nu)$ shall be equal to the sum of two functions, one of μ alone and the other of ν alone. It is evident that the necessary and sufficient condition is

$$D_\nu D_\mu \phi(\mu\nu) = 0. \tag{51}$$

A single term in (51) is $\dfrac{J.\bar{C}\bar{D}}{\Sigma(\mu, \nu)}$, where J and Σ have the following meanings:

$$J = \frac{-3(\mu-\nu)(\varepsilon_3-\varepsilon_4)^2}{2\sqrt{(\mu-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_3)(\nu-\varepsilon_4)}}, \tag{52}$$

$$\begin{aligned}\Sigma(\mu, \nu) = [\bar{A}\sqrt{(\mu-\varepsilon_1)(\nu-\varepsilon_1)} &+ \bar{B}\sqrt{(\mu-\varepsilon_2)(\nu-\varepsilon_2)} \\ &+ \bar{C}\sqrt{(\mu-\varepsilon_3)(\nu-\varepsilon_3)} + \bar{D}\sqrt{(\mu-\varepsilon_4)(\nu-\varepsilon_4)}]^4.\end{aligned} \tag{53}$$

The coefficients of $\bar{A}^2, \bar{B}^2, \bar{C}^2, \bar{D}^2$ in (51) vanish, so that the final form contains only the six terms of the type of $\bar{C}\bar{D}$. These two abbreviations are made:

$$\Gamma(\mu, \nu) = \sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_1)(\nu-\varepsilon_2)(\nu-\varepsilon_3)(\nu-\varepsilon_2)}, \tag{54}$$

$$\begin{aligned}\Phi(\mu, \nu) = {} & \bar{A}\bar{B}(\varepsilon_1-\varepsilon_2)^2\sqrt{(\mu-\varepsilon_3)(\mu-\varepsilon_4)(\nu-\varepsilon_3)(\nu-\varepsilon_4)} \\ & + \bar{A}\bar{C}(\varepsilon_1-\varepsilon_3)^2\sqrt{(\mu-\varepsilon_2)(\mu-\varepsilon_4)(\nu-\varepsilon_2)(\nu-\varepsilon_4)} \\ & + \bar{A}\bar{D}(\varepsilon_1-\varepsilon_4)^2\sqrt{(\mu-\varepsilon_2)(\mu-\varepsilon_3)(\nu-\varepsilon_2)(\nu-\varepsilon_3)} \\ & \pm \bar{B}\bar{C}(\varepsilon_2-\varepsilon_3)^2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_4)(\nu-\varepsilon_1)(\nu-\varepsilon_4)} \\ & + \bar{B}\bar{D}(\varepsilon_2-\varepsilon_4)^2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_3)(\nu-\varepsilon_1)(\nu-\varepsilon_3)} \\ & + \bar{C}\bar{D}(\varepsilon_3-\varepsilon_4)^2\sqrt{(\mu-\varepsilon_1)(\mu-\varepsilon_2)(\nu-\varepsilon_1)(\nu-\varepsilon_2)}.\end{aligned} \tag{55}$$

Equation (51) becomes

$$\frac{(\mu-\nu)\Phi(\mu, \nu)}{\Gamma(\mu, \nu)\Sigma(\mu, \nu)} = 0. \tag{56}$$

The first member of (56) is to vanish for all values of μ and ν, but, from (48), will become infinite for μ or ν equal to $\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4$ unless $\Phi(\mu, \nu)$ shall vanish for these values of μ and ν.

There follow then twelve equations of the type

$$\Phi(\varepsilon_1\varepsilon_2) = 0, \tag{57}$$

from which comes, by reference to (55),

$$\bar{A}\bar{B} = \bar{A}\bar{C} = \bar{A}\bar{D} = \bar{B}\bar{C} = \bar{B}\bar{D} = \bar{C}\bar{D} = 0. \tag{58}$$

Since $\Sigma(\mu, \nu)$ cannot vanish, else ϕ would have no meaning, the following case is excluded,

$$\bar{A} = \bar{B} = \bar{C} = \bar{D} = 0. \tag{59}$$

Equations (58) and (59) give four cases

$$\text{(a)}\quad \bar{A} \neq 0, \quad \bar{B} = \bar{C} = \bar{D} = 0, \tag{60}$$

$$\text{(b)}\quad \bar{B} \neq 0, \quad \bar{C} = \bar{D} = \bar{A} = 0, \tag{61}$$

$$\text{(c)}\quad \bar{C} \neq 0, \quad \bar{D} = \bar{A} = \bar{B} = 0, \tag{62}$$

$$\text{(d)}\quad \bar{D} \neq 0, \quad \bar{A} = \bar{B} = \bar{C} = 0. \tag{63}$$

The conditions given in (60) . . . (63) are not only necessary in order that $D_\nu D_\phi(\mu, \nu)$ may be finite for certain values of μ and ν, but they are also sufficient to make $D_\nu D_\phi(\mu, \nu)$ vanish for all values of μ and ν, as is evident from (55). From (45) . . . (48) the four cases above become

$$\text{(a)}\quad B \neq 0, \quad A = E = B - D = 0, \tag{64}$$

$$\text{(b)}\quad B \neq 0, \quad A = E = B + D = 0, \tag{65}$$

$$\text{(c)}\quad A \neq 0, \quad B = D = A + \rho^2 E = 0, \tag{66}$$

$$\text{(d)}\quad A \neq 0, \quad B = D = A - \rho^2 E = 0. \tag{67}$$

Since A, B, C, D are functions of α, β, γ, δ, it follows that (64) . . . (67) are the conditions under which (37) may exist, and thus make it possible to obtain V in the form desired. In case (a), equation (38), is resolved into the two ordinary differential equations

$$\frac{d^2M}{ds^2} - \left[\frac{(m^2 - \frac{1}{4})\,H.K}{\mu - \varepsilon_1} + n\right] M = 0, \tag{68}$$

$$\frac{d^2N}{dt^2} - \left[\frac{(m^2 - \frac{1}{4})\,H.K}{\nu - \varepsilon_1} + n\right] N = 0. \tag{69}$$

Cases (b)(c)(d) give similar results.

If $E_1(\mu)$, $E_2(\nu)$ be the solutions of any pair of equations of the form of (68)(69), then the value of V is

$$V = \frac{1}{\sqrt{r'}}\left[A \cos m\theta + B \sin m\theta\right] E_1(\mu)\, E_2(\nu). \tag{70}$$

Equation (70) completes the solution of the problem of expressing V in the form of Lamé's Products and an extraneous factor.

Meridian curves corresponding to the four cases.

Having obtained the conditions satisfied by α, β, γ, δ, the nature of the meridian curves corresponding to the four cases may now be determined. Equations (39)(40) express A, B, D, E in terms of α, β, γ, δ and their conjugates.

Separating α, β, γ, δ into real and imaginary parts gives

$$\alpha = a + a_0 i, \quad \alpha_1 = a - a_0 i, \quad \gamma = c + c_0 i, \quad \gamma_1 = c - c_0 i, \tag{71}$$

$$\beta = b + b_0 i, \quad \beta_1 = b - b_0 i, \quad \delta = d + d_0 i, \quad \delta_1 = d - d_0 i. \tag{72}$$

It is assumed that the determinant of the linear transformation does not vanish, i. e.

$$\alpha\delta - \beta\gamma \neq 0. \tag{73}$$

It follows from (71)(72) that the two following expressions cannot vanish simultaneously:

$$ad - bc + b_0 c_0 - a_0 d_0, \tag{74}$$

$$a_0 d - b_0 c - b c_0 + a d_0 . \tag{75}$$

From (64)(71) . . . (75) the following results for case (a) are obtained:

$$\frac{a}{a_0} = \frac{-b_0}{b} = \frac{c}{c_0} = \frac{-d_0}{d} = m \; (m \text{ real}, \neq 0), \tag{76}$$

$$x' + r'i = \frac{a + b_0 i\,(x + ri)}{c + d_0 i\,(x + ri)}. \tag{77}$$

Similarly for case (b),

$$\frac{a}{a_0} = \frac{b}{b_0} = \frac{c}{c_0} = \frac{d}{d_0} = n \; (n \text{ real}, \neq 0), \tag{78}$$

$$x' + r'i = \frac{a + b\,(x + ri)}{c + d\,(x + ri)}. \tag{79}$$

An analysis of (77) will show that the surfaces of the system may be obtained from the curves of (1) and (2) by subjecting them to an inversion with regard to a point on the r axis, followed by a translation along that axis and then using it as the axis of rotation. Similarly from (79) it may be shown that the centre of inversion is on the x axis, and the same axis is the axis of revolution. For cases (c) and (d) the equations found are

$$ad + a_0d_0 - bc - b_0c_0 = 0, \tag{80}$$

$$a_0d - ad_0 + b_0c - bc_0 = 0, \tag{81}$$

$$a_0c - ac_0 \pm \rho^2 (b_0d - bd_0) = 0, \tag{82}$$

$$\pm \rho^2 = \frac{a^2 + a_0^2}{b^2 + b_0^2} = \frac{c^2 + c_0^2}{d^2 + d_0^2}. \tag{83}$$

In (82)(83) the upper signs belong to case (a), the lower signs to case (b).

From (83), case (c) exists only when ρ^2 is a positive real quantity. The transformation may be followed by considering its effect on the circle whose radius is ρ and centre is at the origin. The equation of this circle may be written, with ϕ as a variable angle,

$$z = \rho (\cos \phi + i \sin \phi). \tag{84}$$

The transformation is

$$x' + r'i = \frac{\alpha + \beta z}{\gamma + \delta z}, \tag{85}$$

where α, β, γ, δ are subject to the conditions of (80) . . . (82). From (84)(85),

$$x' + ir' = \frac{P + Qi}{S}, \tag{86}$$

where P, Q, S are real and Q has the value

$$\begin{aligned} Q = {} & ca_0 - ac_0 + \rho^2 (b_0d - bd_0) \\ & + \rho \sin \phi (bc - ad + b_0c_0 - a_0d_0) + \rho \cos \phi (b_0c - bc_0 + a_0d - ad_0). \end{aligned} \tag{87}$$

But (80) . . . (82) make Q become zero, and (86) gives

$$r' = 0. \tag{88}$$

It follows that in case (c), provided ρ^2 is real and positive, the circle whose radius is ρ is transformed into the axis of revolution. This circle is a circle of symmetry, as may be shown from (1) and (2).

A similar treatment of case (d) shows that it exists only when ρ^2 is a negative real quantity, and that the circle of symmetry whose radius is $\pm \rho i$ is transformed into the axis of revolution.

The four cases may be stated briefly as follows: In case (a) the r axis becomes the axis of revolution, while in case (b) the axis of revolution is unchanged.

In cases (c) and (d) the real circle of symmetry, if such exists, becomes the axis of revolution.

In all cases an inversion may take place with regard to a point on the axis of revolution, followed by a translation along that axis, this inversion of the curves resulting in a space inversion of the surfaces.

The real surfaces obtained.

The meridian curves of (1) and (2) are real if ρ^2, ε_1, ε_2, ε_3, ε_4 are real, also if ρ^2 is pure imaginary, while ε_1, ε_2 are real and ε_3, ε_4 conjugate imaginaries. In the first instance the curves are those which Holzmüller* calls sn. curves, and in the second instance cn. curves, convenient designations which will be retained in the following pages. The transformations in cases (a)(b)(c)(d) are applicable to the sn. curves, while cases (a) and (b) only are applicable to the cn. curves.

The four cases give no new surfaces, for all the surfaces obtained by them may be obtained by the following method: If (1) and (2) represent the sn. curves, then an interchange of ε_1, ε_2, ε_3, ε_4, followed by inversion with regard to a point on the axis of revolution, and a translation along that axis, gives all the surfaces of cases (a)(b)(c)(d).

If the equations represent the cn. curves, then ε_1 and ε_2 may be interchanged, followed by inversion and translation as before, giving the surfaces of cases (a) and (b).

If the surfaces are not real, then cases (a) and (b) only exist, and one or the other of the coördinate axes is the axis of revolution, admitting inversion and translation as before.

Confocal spherical cyclic curves.

Confocal spherical cyclic curves may be defined as the curves on the surface of a sphere obtained by inversion from the general confocal plane cyclic curves.

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*Isogonale Verwandschaften, Holzmüller, S. 256.

It appears from the work of the previous section that if the most general family for which it has now been found that Lamé's Products, with the extraneous factor $\frac{1}{\sqrt{r}}$ can be obtained, be subjected to the inverse transformation, changing meridian planes into concentric spheres, the result is merely confocal cones of the second degree. It will now be shown that the general family of spherical cyclic curves goes over, by the direct transformation—equations (32) . . . (36), Section I—into a family of confocal plane cyclic curves.

The detailed transformations of the previous paper show that one sphere of the family of concentric spheres becomes a plane after the first inversion.

In order to prove the proposition above, consider the general unsymmetrical cyclic curves and let the plane of these curves be the plane derived from a certain sphere of a family of concentric spheres by a portion of what has been called the direct transformation. If the inverse of this transformation be now applied to the plane of the curves, the result is the original sphere, and the general family of confocal spherical cyclic curves.

In applying the transformation which will carry concentric spheres into meridian planes, the first portion will merely reproduce the original curves, while the remainder will be essentially an inversion of the plane of the curves. Consequently the curves finally obtained are plane confocal cyclic curves, for they are the result of a linear transformation of the original curves.

Since it has been shown in this paper that Lamé's Products, with the extraneous factor $\frac{1}{\sqrt{r}}$, can be found only when the cyclic curves in the meridian planes have the axis of revolution as an axis of symmetry, it follows that for a system of conical coördinates in which confocal cyclic curves are cut out on concentric spheres, Lamé's Products can be found only when the cones are of the second order.

Articles by Wangerin and Haentzschel.

In addition to the article previously mentioned, Wangerin has written another article* (treating of surfaces of revolution) in which the extraneous

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***Reduction der Potentialgleichung, Wangerin. Monatsberichte der Kgl. Akademie der Wissenschaften zu Berlin, 1878.**

factor is at first undetermined. He obtains the sufficient conditions under which (37) may exist, and also obtains $\frac{1}{\sqrt{r}}$ as the form of the extraneous factor. That his conditions are also necessary follows from a discussion by Boehm.*

Corresponding to these conditions, Wangerin obtains surfaces of the fourth degree whose meridian curves are the sn. and cn. curves. These curves may be obtained by equating the real and imaginary parts of the following equation:

$$x + ri = f(t + m), \tag{89}$$

where f is either sn. or cn.

The curves obtained by using tn. or dn. are of the same form as the sn. curves.

Wangerin states that the most general surfaces of revolution for which Lamé's Products, with an extraneous factor, exist, are those whose meridian curves are obtained from the curves above by an inversion with respect to a point on the axis of revolution. These surfaces may be shown to be identical with those obtained in the preceding pages.

Haentzschel† treats some of the same topics as Wangerin but obtains surfaces of revolution of the thirty-second degree, which will not be considered here. He also treats at some length a special form of his most general solution, viz. a linear transformation of the Weierstrass $\wp$ function, where the coefficients are subject to restrictions. (Equation (89) with f written as $\wp$ gives the fundamental curves.)

The resulting surfaces of the fourth degree are identical with those of the present paper. He also obtains surfaces which he states to be of the eighth degree, but these are in fact of only the fourth degree, and of the same type as above.

He has overlooked the fact that the restrictions which he mentions will give a zero value to several coefficients in equations (31)(32)(36) on page 23 of his book.

HARVARD UNIVERSITY, *Dec.*, 1897.

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*Boehm, Reduction Partieller Differentialgleichungen, Leipzig, 1896.

†Reduction der Potentialgleichung, E. Haentzschel, Berlin, 1893.